# Meeting 1: Nash Equilibrium

**Advisors:** ***Ioannis Panageas, Emmanouil Vlatakis*** **Student:** ***Fivos Kalogiannis***

## Objectives

- Define Sperner's Lemma for the multi-dimensional case
- Prove the Brouwer Fixed Point Theorem (consider Sperner's lemma given)
- Prove the $1^{st}$ Theorem from Nash (1951)

## 1.1 Preliminaries

In order to start talking about equilibrium points in games we need to familiarize ourselves with a couple of tools. In order to prove the existence of an equilibrium point in a finite $N$-person game we will need the Brouwer Fixed Point Theorem which in turn needs Sperner's Lemma.

We will use an analogy from a common practice in building wire-frames in order to get into graph theory gently. For this purpose a graph's edge will be represented by a stick and a joint made out of playdough will stand in the place of a graph's vertex.

### 1.1.1 On Sticks, Colored Playdough, and Triangles (Sperner's Lemma)

#### 1.1.1.1 One dimension and Two dimensions

We are given $n$ buckets of playdough – each bucket's content is colored uniquely – and as many identical sticks as we like. Using playdough to construct joints we connect sticks together. The joints are what we refer to as nodes or vertices in graph theory and the sticks are called edges. The color of the playdough will remain called as such.

Let us imagine a simple recipe for constructing however large structures starting from a single dimension structure. Consider a line consisting by a series of edges connected consecutively by edges. We pick 2 different colors to label with the rightmost and the leftmost vertices respectively. Now, we can color the rest of the vertices using the same 2 colors.

**Lemma 1** (Sperner's lemma in a single dimension)**.** Given a set of vertices and edges that alternate forming a straight line with one vertex on each end of the line and two distinct colors; if the two ends are colored differently and the intermediate vertices can be colored only by the same two colors used for the two ends, then the number of color changes between the vertices of line will be an odd number.

Of course, the latter are trivial. We shall now state the rule for building greater structures as promised.

**Top-Down Approach** Form a large triangle consisting of 3 big sticks and 3 distinctly colored pieces of playdough. We will now break every edge to as many pieces we may in order to be able to create more connections within the initial triangle. But there are some rules to that; every time you break one of the exterior edges you have to stick them back together using playdough of the same color found on its ends. Also, every time you break one of the exterior sticks you have to make sure that no other polygon than a triangle is formed, you will have to break any other edge you find necessary while respecting the rule that every node of each side of the exterior triangle is colored with either one of the colors found on the two respective ends.

The top-down approach may – honestly – seem a bit cluttered, so let's try a bottom-up one.

**Bottom-Up Approach** Consider new bucket of playdough. The playdough is colored gray and we will consider it as colorless and a placeholder for replacing it later with some that is colored. We onstruct a simple triangle with 3 sticks and gray playdough. We start building up from the initial simple triangle by making sure the outer shape remains a triangle by either adding two new sticks and a new joint. Whenever we may and if – as required – the outer edges still form a triangle you we stop the building phase and start the coloring phase. For the coloring phase we pick a unique color for every on of the vertices of the outermost triangle. We color every joint of the outside triangle – color as in replace the gray plaster with a colored one – with strictly either one of the colors found on one of the two ends. The interior vertices need to be colored as well, but we can do so randomly.

We remind the reader that such a wire-frame made of playdough and sticks is a graph $G(V, E)$, whose vertices $V$ are the joints made of playdough and its $E$ are the sticks we used. The color of the playdough is a label we assign to every vertex $v \in V$. The colors can be equivalently thought of as an integer.

**Definition 1.** *A triangle is called distinguished if its vertices are colored by all 3 colors.*

**Lemma 2** (Sperner's lemma on a planar graph)**.** Every properly colored triangular subdivision (or triangulation) of a triangle has an odd number of 3-colored triangles.

#### 1.1.1.2 More than two dimensions

In order move on two more than two dimensions one should seek for a generalizations of the notion of a triangle. The generalization in question is a simplex. In the 0-dimensional case a simplex reduces to a point, in the 1-dimensional case a simplex reduces to a line, in the 2-dimensional case a simplex is a triangle, in the 3-dimensional case a simplex is a tetrahedron (a pyramid whose every side is a triangle) and lastly a 4-dimensional space is called a 5-cell (as it consists of 5 tetrahedra). Some authors symbolize $m$-dimensional simplices (or $m$-simplices) with $\Delta^m$.

**Definition 2** ($m$-simplex)**.** *Consider an $m$-dimensional space and vectors $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ drawn from the former space. If the vectors $\boldsymbol{v}_1 - \boldsymbol{v}_0, \boldsymbol{v}_2 - \boldsymbol{v}_0, \ldots, \boldsymbol{v}_m - \boldsymbol{v}_0$ are linearly independent, the convex hull of these vectors (namely, the intersection of all convex sets that contain the points $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_m$) is said to be an $m$-simplex, $\Delta^m$.*

It may have become apparent that we can speak of the vectors $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ as the vertices of the simplex as well.

**Definition 3.** *A simplex $P$ is called a* ***probability simplex*** *or a* ***standard simplex*** *if*

$$P = \{x = \theta_0 \boldsymbol{v}_0 + \ldots + \theta_m \boldsymbol{v}_m : \theta_i \geq 0, \sum_{i=0}^{m} \theta_i = 1\}$$

.

That means that all points of $P$ lie in the convex set defined by the origin $\mathbf{0}$ and $m$ unit vectors of $\mathbb{R}^m$.

**Definition 4** ($k$-face)**.** *A $k$-face of an $m$-simplex, is merely a $k$-simplex defined by a subset of the $m+1$ vertices of the initial $m$-simplex such that $k \leq m$.*

We name $P$ the $m$-simplex defined by $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$.

**Sperner coloring on an $m$-simplex** Let $\mathcal{S}$ be a set of vertices and edges lying inside an $m$-simplex including vertices and the edges of the $m$-simplex itself. The points of $\mathcal{S}$ define smaller $m$-simplices that divide the larger simplex into disjoint spaces. $\mathcal{S}$ is also called a triangulation of $P$. Keeping in mind that colors are refered to uniquley as color $0, 1, 2, \ldots, m$, a valid Sperner coloring is a coloring of every vertex for which the following hold:

- Every one of the $m+1$ vertices that define the $m$-simplex get a unique color 1 through $m+1$. (With no loss of generality and for the sake of clarity, the vertex $v_i$ gets the $i$-th color.)

- Vertices of $\mathcal{S}$ that lie on a $k$-simplex defined by a subset of $\{v_0, \dots, v_m\}$ can only receive colors whose numbers coincide with the indices of the $k+1$ vertices that define the $k$-simplex in question. (e.g. vertices lying on a 4-simplex defined by $\{v_0, v_2, v_4, v_5\}$ can only be colored with colors $\{0, 2, 4, 5\}$)

**Lemma 3** (Sperner's lemma on an $m$-simplex)**.** Given an $m$-simplex $P$, a division $\mathcal{S}$ of $P$ that retains a valid Sperner coloring with $m+1$ colors, there exists an odd number of $m$-simplices inside $P$ that carry all $m+1$ colors.

### 1.1.2 Brouwer Fixed Point Theorem

**Theorem 1** (Brouwer Fixed Point Theorem)**.** Let a set $K \subseteq \mathbb{R}^d$ be convex, closed and bounded. A function $\boldsymbol{T} : K \to K$ that is continuous has at least one $\boldsymbol{x} \in K$ such that $\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{x}$.

We prove the Brouwer theorem on $K = \Delta^m$ by means of the Sperner Lemma. We show that there is a relation between entries of $\boldsymbol{Q} \in K$ and $\boldsymbol{T}(\boldsymbol{Q})$ that can be considered as a valid Sperner coloring. We will talk of $k$-faces of $\Delta^m$ as being constituted by k vectors $v_r$ vectors or the indices of these vectors interchangeably.

*Proof.* We consider the case that $K = \Delta^m$. Since every $\boldsymbol{Q} = (q_1, \dots, q_m) \in K$ and $\boldsymbol{T}(\boldsymbol{Q}) = (T_1(\boldsymbol{Q}), \dots, T_m(\boldsymbol{Q})) \in K$ we know that:

$$\sum_i q_i = \sum_i T_i(\boldsymbol{Q}) = 1$$

By means of the pigeon hole principle we can conclude that there exists at least one index $j$ such that $q_j \geq T_j(\boldsymbol{Q})$.

If $\boldsymbol{Q}$ happens to lie on a $k$-face of the larger $\Delta^m$ simplex, we know that there are several non-zero $q_i \geq 0$ with $i \in I$, where $I$ signifies the indices of the vectors $v_r$ that constitute the $k$-face in question. Apparently, for the rest $q_i$ s.t. $i \notin I$, $q_i = 0$.

Combining the two latter propositions, since the entries $q_i$ that refer to the vertices perpendicular to the current $k$-face are zero, we can pick a $j$ s.t. $q_j \geq T_j(\boldsymbol{Q})$ from only the set of indices that constitute the current $k$-face.

Picking an index from those that correspond to the set of vectors that construct a $k$-face is directly equivalent to picking a color that is compatible to a Sperner coloring – remember that a vertex's color can only come from the set of colors of the vertices that constitute the $k$-face in question.

We firstly use this rule on the vertices of $\Delta^m$ as to assign a number 0 through $m+1$ to each color.

Picking arbitrarily many points $Q_i$ we can create an arbitrarily small triangulation $\mathcal{S}$ of $P$ that has a valid Sperner coloring. Thanks to the Sperner lemma for the multi-dimensional case, we know that there is an odd number of cells (smaller $m$-simplices) of $\Delta^m$ such that every vertex is uniquely colored. But, having a cell being uniquely colored means that for every vertex $Q^{(r)}, r \in \{0, 1, \dots m\}$:

$$q_i^{(r)} \geq T_i(\boldsymbol{Q}^{(r)})$$

If all $Q^{(r)}$ can come infinitesimally close to each other depending on the number of initial points in $\mathcal{S}$, we can conclude that there will be such a point $\boldsymbol{Q}^* = (q_1^*, \dots, q_m^*)$, infinitesimally close to the points $\boldsymbol{Q}^{(r)}$ of the distinguished simplex that since $\sum_i q_i^* = 1 = \sum_i T_i(\boldsymbol{Q}^*)$ and $q_i \geq T_i(\boldsymbol{Q}^*)$ it must inevitably hold that $\boldsymbol{Q}^* = \boldsymbol{T}(\boldsymbol{Q}^*)$ $\square$

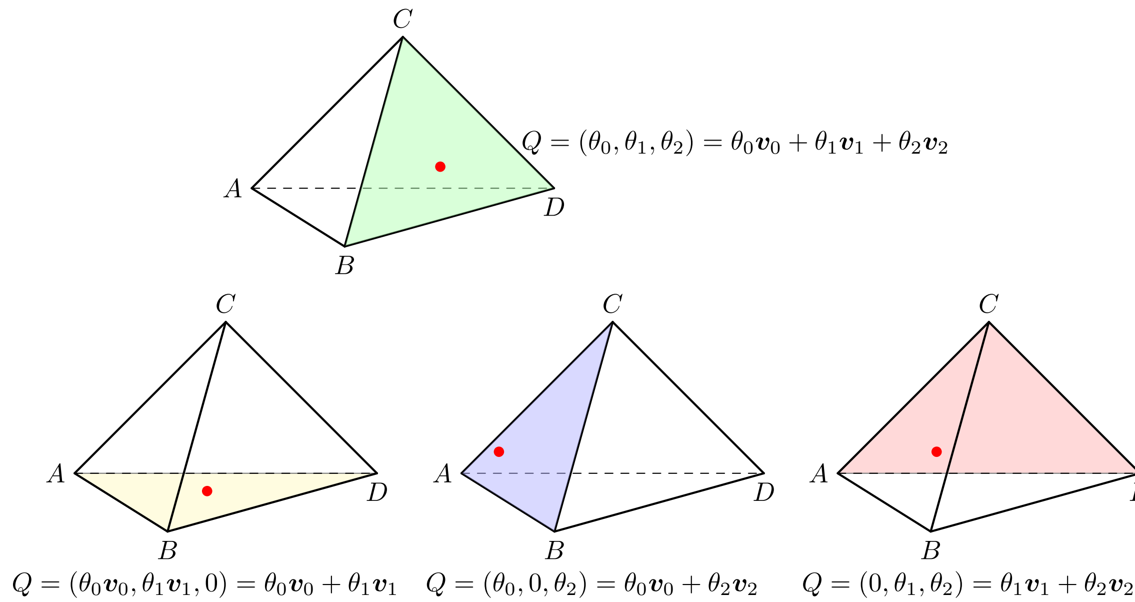


Figure 1.1: The 3-faces of $\Delta^3$ with their corresponding coordinates

## 1.2 Nash Equilibrium

**Rock-Paper-Scissors** Rock-paper-scissors is a game played between two players that lasts only one round. Each player has a selection of 3 moves (namely rock, paper and scissors). Each player picks their move and simultaneously reveal their choice. A player can win or lose and there can be only one winner.

In terms conventional to game theory:

- this is a 2-person game
- each one of the 3 moves is a *(pure) strategy* and can be thought of as three perpendicular unit vectors, $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$. (More precisely strategy is the player's all-or-nothing bet that the corresponding move will win)
- The rule by which we decide which of the played moves wins can either be formulated as a matrix and in that case the matrix would be called a *payoff matrix*, or it could be thought of as a function, in which case it would be called a *payoff function*
- the fact that only one player can be a winner for every instance of the game makes the game a *zero-sum game*. Think about winning as having payoff 1 and losing as $-1$, in that case the sum of every player's payoffs equals to 0

**A more general case** The latter may not always hold, in the sense that:

- a game may have more than two players. Such a game is obviously called a $n$-player game
- a player need not go *all-in* on each move but can rather place partial bets on several ones. Such a betting is called a *mixed strategy.*
- There can be $\binom{n}{2}$ payoff matrices, one for every player that will relate a player's bet to their payoff with regards to every other player's bets
- it may not always be true that the sum of the payoffs is equal to zero. Such games are called *general-sum games*

Just to be clear, a lowercase Latin letter (e.g. $i, j, k$) signifies one out of $n$ players and a lowercase Greek letter signifies one of $m$ moves. Each player's strategy can be represented by a vector $\boldsymbol{x}_i = (x_{i\alpha}, x_{i\beta}, \ldots, x_{i\kappa}, \ldots) \in \Delta^{m-1} \subset \mathbb{R}^d$ whose every entry $i$ holds the player's betting on the $i$-th move. It may be obvious that we consider that $\sum_\lambda x_{i\lambda} = 1, x_{i\lambda} \geq 0$. More formally:

**Definition 5** (Finite Game)**.** [1] *A finite $n$-person game is:*

- *a **general-sum game***
- *consisting of $n$ players*
- *each player $i$ is associated with an $m$-dimensional **strategy vector** $\boldsymbol{x}_i \in \Delta^m$. A **mixed strategy** vector is a vector whose entries sum to one. **Pure strategies** (i.e. $\boldsymbol{x}_i = \boldsymbol{e}_{i\kappa}$ for some $\kappa \in \{1, \ldots, m\}$) are special cases of mixed strategies.*
- *each player $i$ is associated with a **payoff function** $p_i : \underbrace{\Delta^m \times \cdots \times \Delta^m}_{n} \to [0, 1]$. Payoff, the scalar value of a payoff function, could be referred to as **utility** as well.*

We will also define some auxiliary observations and notations; the $\kappa$-th pure strategy of the $i$-th player will be denoted by $\boldsymbol{e}_{i\kappa}$ and a mixed strategy $\boldsymbol{x}_i = (x_1, \ldots, x_m)$ can equivalently be written as $\sum_\kappa x_{i\kappa}\boldsymbol{e}_{i\kappa}$. Also, $\boldsymbol{s}$ will be the tuple of all mixed strategies $(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m)$ and a swapping of $i$-th player's strategy vector $\boldsymbol{x}_i$ with a new one $\boldsymbol{t}_i$ will conveniently be noted as $(\boldsymbol{s}; \boldsymbol{t}_i)$. Successive substitutions $((\boldsymbol{s}; \boldsymbol{t}_i); \boldsymbol{r}_j)$ will be indicated as $(\boldsymbol{s}; \boldsymbol{t}_i; \boldsymbol{r}_j)$. A pure strategy $\kappa$ is said to be *used* by player $i$ when the coefficient $x_{i\kappa} > 0$. We also define $p_{i\kappa}$ as replacing player $i$'s mixed strategy with the pure strategy $e_{i\kappa}$.

**Definition 6** (Equilibrium Point)**.** *A tuple of mixed strategies $\boldsymbol{s} = (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$ is said to be an equilibrium point if for every $i$:*

$$p_i(\boldsymbol{s}) = \max_{\forall \boldsymbol{r}_i \in \Delta^m} p_i(\boldsymbol{s}; \boldsymbol{r}_i)$$

In plain English, the latter means that an equilibrium point is an $n$-tuple of mixed strategies such that no player can unilaterally (i.e. by themselves) increase their payoff if the strategies of all other players are held fixed.

**Theorem 2** (Existence of Equilibrium Points)**.** Every finite general-sum game has an equilibrium point.

*Proof.* The proof we selected relies on the Brouwer Fixed Point Theorem. We will define a function $\boldsymbol{T} : \boldsymbol{s} \to \boldsymbol{s}'$, that maps a strategy $\boldsymbol{s}$ to a new one $\boldsymbol{s}'$.

First off, we observe that $p_i$ is linear with respect to every strategy vector $\boldsymbol{x}_i$ and $n$-linear with respect to all $n$ of them. Consequently:

$$\max_{\forall \boldsymbol{r}_i \in \Delta^m} p_i(\boldsymbol{s}; \boldsymbol{r}_i) = \max_\kappa p_i(\boldsymbol{s}; \boldsymbol{e}_{i\kappa})$$

Then we let a function $\phi_{i\kappa}$ be a *continuous* function used as an indicator of how much would a player's payoff would improve if they were to play a pure strategy $\kappa$ instead (deteriorating is as good as no change at all):

$$\phi_{i\kappa} = \max\{0, p_{i\kappa}(\boldsymbol{s}) - p_i(\boldsymbol{s})\}$$

With the help of the $\phi_i$ functions we move on to define a function $\boldsymbol{T}(\boldsymbol{s}) = \boldsymbol{s}'$ that will perform an one-step payoff increase for every player's strategy (if no increase is possible, payoff stays the same). The function in question can be seen as a two step process; namely it performs a step in every dimension from $x_{i\kappa}$ towards the pure strategy that improves the payoff for the given player and then a scaling of every dimension in order to keep the sum of all the coefficients $\sum_\kappa x_{i\kappa} = 1$. So for every strategy $\boldsymbol{x}$, function $\boldsymbol{T}$ outputs a new one $\boldsymbol{x}'$ s.t.:

$$x_i' = \overbrace{(x_i + \sum \phi_{i\kappa}\boldsymbol{e}_{i\kappa})}^{step} \cdot \overbrace{\frac{1}{1 + \sum_\kappa \phi_{i\kappa}}}^{scaling}$$

[1] In a more general case, not all players need to have the same number of moves but this will not concern us at the moment and we will not lose any of the generality from the conclusions we are going to draw.

If $\boldsymbol{s}' = \boldsymbol{s}$ stays the same under $\boldsymbol{T}$ it means that no player can shift to a strategy that will make their payoff better which makes $\boldsymbol{s}$ an equilibrium point. Conversely, if $\boldsymbol{s}$ is an equilibrium point we need to observe that all $\phi.(\cdot)$ turn to zero which makes $x_i' = (x_i + 0)\frac{1}{1} = x_i$

Since $\boldsymbol{T}$ is continuous, bounded and closed on $S = \underbrace{\Delta^m \times \cdots \times \Delta^m}_{n}$ and $S$ is convex, thanks to the Brouwer Fixed Point Theorem, we can decide that there exists an equilibrium point, or $\boldsymbol{T}(\boldsymbol{x}) = \boldsymbol{x}$ for every finite game. $\square$

## References

Karlin, A. and Peres, Y. (2017). *Game Theory, Alive*. American Mathematical Society. URL http://www.ams.org/mbk/101

Nash, J. (1951). Non-cooperative games. *Annals of mathematics* 286–295.